David Kruzel 12 July 2024

Calculating Boninger's Polynomials

References.

- Joe Boninger, The Jones Polynomial from a Goeritz Matrix
- Virtual Knot Theory, Kauffman
- Thistlethwaite, A Spanning Tree Expansion of the Jones Polynomial

Thistlethwaite's τ polynomial

A signed graph is a graph G equipped with a sign function $\varepsilon : E(G) \longrightarrow \{+1, -1\}$

For any signed graph G, $\tau(G)$ is a polynomial in A defined via the following contraction - deletion relations:

- if $e \in E(G)$ is a loop then
$$\tau(G) = -A^{3\varepsilon(e)}\tau(G-e)$$
- if $e \in E(G)$ is a bridge then
$$\tau(G) = -A^{-3\varepsilon(e)}\tau(G/e)$$
- if $e \in E(G)$ is neither a loop nor a bridge then
$$\tau(G) = A^{\varepsilon(G)}\tau(G/e) + A^{-\varepsilon(G)}\tau(G-e)$$

We also put $\tau(\bullet) = 1$. Today we will only focus on connected graphs.

Boninger's ν polynomials

When Boninger introduced his polynomials he did not have virtual knots in mind. For the sake of today's discussion, we will be focusing on virtual knots.

Given a checkerboard-colorable virtual knot K, we can take τ of the Tait graphs of K. For a Tait graph Γ of K, we define

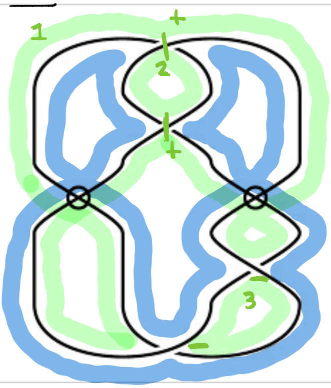
$$\nu_{K,\Gamma}(t) = \left[(-A)^{-3w(K)}\,\tau(\Gamma)\right]$$

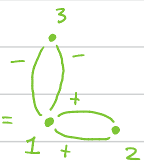
where $w(K)$ = writhe of K.

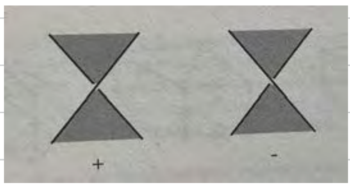
Some Examples

- For any classical knot K with Tait graphs Γ, Γ', we have $\nu_{K,\Gamma}(t) = \nu_{K,\Gamma'}(t) = J_K(t)$ = Jones polynomial of K.

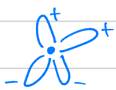
4.99



$\Gamma =$ 



$$\tau(\Gamma) = A^{8} + 2 + A^{-8} = (-A^{4} - A^{-4})^{2}$$

$\Gamma' =$ 

$$\tau(\Gamma) = (-A^{4} - A^{-4})^{2}\,\tau(\Gamma')$$

$$\tau(\Gamma') = 1$$

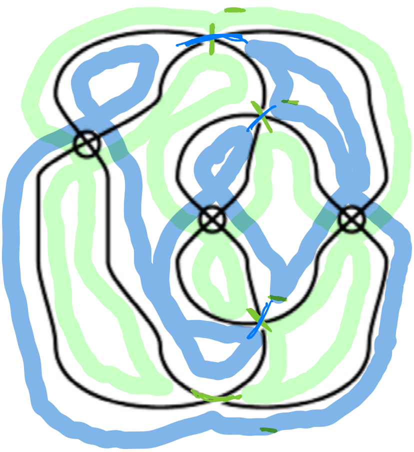
$\Gamma =$ 

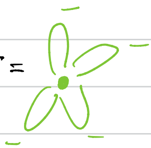
$$\tau(\Gamma) = -A^3$$

$\Gamma' =$ 

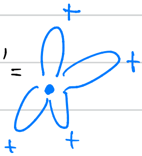
$$\tau(\Gamma') = A^{-1} + A^7$$
$$= (-A^3)(-A^4 - A^{-4})$$
$$= (-A^4 - A^{-4})\,\tau(\Gamma)$$

Virtual Knot 4.107



$\Gamma =$ 

$$\tau(\Gamma) = A^{-12}$$

$\Gamma' =$ 

$$\tau(\Gamma') = A^{12}$$

Gauss Codes

A Gauss Code is a way of representing knot diagrams.

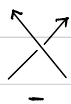
For us, a Gauss Code with n classical crossings is some ordering of the set $\{1, -1, 2, -2, \ldots, n, -n\}$ along with some length n sequence of +'s and −'s.

For example: $\underbrace{(-1, 2, -3, 1, -2, 3)}$, $\underbrace{(-, -, -)}$

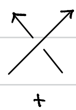
indicate the order of the crossings as we move around the knot

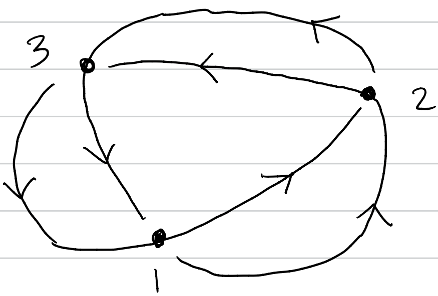
+/− indicate over/under

a +/− in the k^{th} position indicates that the k^{th} crossing looks like



or

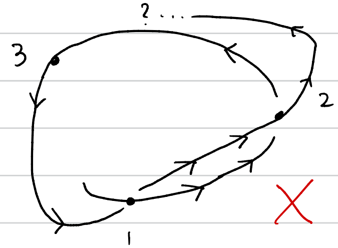




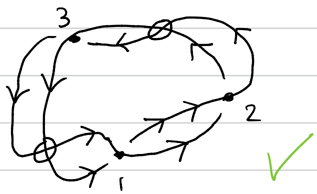
Not all Gauss Codes correspond to planar graphs.

Eg. $(1, -2, 3, \underline{-1}, \underline{2}, -3)$, $(\underline{+}, \underline{+}, -)$

<u>without allowing virtual crossings</u>



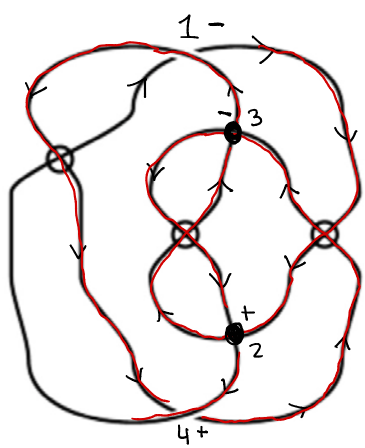
<u>with virtual crossings</u>



A virtual Knot is an equivalence class of Gauss codes under virtual Reidemeister moves. Each possible Gauss code corresponds to a unique Virtual Knot, So when using a computer a computer to compute things we may plug in our Gauss-codes worry free.

Eg.

Virtual Knot 4.107



$(-1, 2, -3, 1, -4, 3, -2, 4)$

$(-, +, -, +)$

Calculating ν

There are a few steps:

- Determine whether or not a given Gauss Code corresponds to a connected, checkerboard colorable diagram
- From the Gauss code, obtain the Tait graphs
- From the Tait graphs, calculate τ

Eg. Calculate $\nu_{K,\Gamma}$, $\nu_{K,\Gamma'}$ for 4.107